GENERATING JACOBI FORMS

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ABSTRACT. In this paper we explore the relationship between vector-valued modular forms and Jacobi forms and give explicit relations over various congruence subgroups. The main result is that a Jacobi form of square-free index on the full Jacobi group is uniquely determined by any of the associated vector components. In addition, an explicit construction is given to determine the other vector components from this single component. In other words, we give an explicit construction of a Jacobi form from a subset of its Fourier coefficients. This leads to results about how the transformation properties are affected by congruence restrictions on the Fourier expansion.

1. Introduction. The relationship between classical Jacobi forms and vector-valued modular forms is well known, see [2] and [12] for example. In an earlier paper [11] it was shown that when certain components of a vector-valued modular form associated to a Jacobi form are zero then some of the others must be zero as well. It was shown that if the index of the Jacobi form is prime, then the Jacobi form is entirely determined by any of the associated vector components, and an explicit algorithm was given for how to generate the rest of the Jacobi form from the single vector component. In this work we look to extend these results to general square-free indexes by giving a construction of a Jacobi form from a single vector-valued function component. In the process of the construction we will discuss uniqueness and the minimal number of components necessary to completely specify a Jacobi form of arbitrary index.

These vector-valued modular forms may be thought of as unitary representations of a metaplectic cover of $Sl_2(Z)$ and we will compute how the representations arise and how the different components interact over varying subgroups. In particular, the representation that arises will be trivial on $\Gamma(4m)$ (where $m$ is the index of the Jacobi form) and we will compute the representation on $\Gamma_1$ and $\Gamma_0$-type groups between $\Gamma(4m)$ and $Sl_2(Z)$. Thus we decompose the representation (and the

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vector-valued modular form) over these subgroups. This will decompose the Jacobi form over subgroups of $SL_2(\mathbb{Z})$ that contain $\Gamma(4m)$. The results in this paper allow one to construct new homomorphisms from certain spaces of half-integral weight modular forms to Jacobi forms. In addition, one can study the restrictions on the possible Fourier expansions for Jacobi forms. The Fourier expansions of a Jacobi form are limited because of the invariance properties of the function. Further, given parts of the Fourier expansion of a Jacobi form, one can use the invariance properties to determine the rest of the expansion. It should be noted that many of these results are implied by the work of Skoruppa [12], however in this paper we seek to give a new derivation of these results and give more explicit methods for moving between the vector-valued modular forms and the Jacobi forms. In particular, by considering the representations directly instead of the associated characters one can study the dependencies among the vector-valued components. This in turn gives more explicit methods for moving between the different spaces of half-integral modular forms. Further, one can use these techniques to study new relations between the Fourier coefficients of a Jacobi form.

2. Basics. In this section we list the specific tools that will be used in the construction (all of these results can be found in [2]). A Jacobi form of weight $k$ and index $m$ (both positive integers) is an analytic function of two variables $f(\tau, z) : \mathfrak{h} \times \mathbb{C} \rightarrow \mathbb{C}$, (where $\mathfrak{h}$ is the complex upper half plane) that satisfies

\begin{equation}
\label{eq:1}
f \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k e^{\frac{2\pi imc z^2}{c\tau + d}} f(\tau, z),
\end{equation}

\begin{equation}
\label{eq:2}
f(\tau, z + \lambda \tau + \mu) = e^{-2\pi i m (\lambda^2 \tau + 2\lambda \mu)} f(\tau, z),
\end{equation}

for all \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\) in $SL_2(\mathbb{Z})$ (or some subgroup) and $[\lambda, \mu]$ in $\mathbb{Z}^2$ and has a Fourier expansion of the form

\begin{equation}
\label{eq:3}
f(\tau, z) = \sum_{n \geq 0} \sum_{\substack{r \in \mathbb{Z} \\ 4m r^2 \geq 0}} c(n, r) q^n \xi^r \text{ where } q = e^{2\pi i \tau}, \xi = e^{2\pi i z}.
\end{equation}

The group that acts on the space $\mathfrak{h} \times \mathbb{C}$ is actually the semi-direct product $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ but this will not be important in what follows so we do not expand upon it here. Throughout the paper we will assume $k$ is the weight and $m$ is the index of the Jacobi form under consideration.

The second invariance formula (2) gives that $c(n, r) = c(n', r')$ whenever $r \equiv r' \mod 2m$ and $4nm - r^2 = 4n'm - (r')^2$. So the coefficients may be written $c_\mu(N) = c(n, r)$ where $N = 4nm - r^2$, $r \equiv \mu$ modulo
2m, and \( c_\mu(N) = 0 \) unless \( N \equiv -\mu^2 \) modulo \( 2m \). The Jacobi form \( f \) may then be expressed as

\[
f(\tau, z) = \sum_{\mu \mod 2m} h_\mu(\tau) \theta_{\mu,m}(\tau, z),
\]

where the \( \theta_{\mu,m} \) are fixed theta functions (depending only on the index)

\[
\theta_{\mu,m}(\tau, z) = \sum_{n \in \mathbb{Z}, n \equiv \mu \mod 2m} q^{n^2/4m} \xi_n,
\]

and the functions \( h_\mu(\tau) \) are given by

\[
h_\mu(\tau) = \sum_{N \geq 0} c_\mu(N) q^{N/\mu}.
\]

These theta functions satisfy (for typesetting purposes we use the standard convention that \( \mu \) \((2m)\) means all \( \mu \) modulo \( 2m \), and \( e_N(M) = e^{2\pi i M/N} \))

\[
(\theta_{\mu,m}(\tau + 1, z))_{\mu \in \mathbb{Z}} = \text{Diag} \left( e_{4m}(\mu^2) \right)_{\mu \in \mathbb{Z}} \left( \theta_{\mu,m}(\tau, z) \right)_{\mu \in \mathbb{Z}},
\]

\[
\left( \theta_{\mu,m} \left( \frac{-1}{\tau}, \frac{z}{\tau} \right) \right)_{\mu \in \mathbb{Z}} = \sqrt{2m} e^{-2\pi i m z^2} \tau^m \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \theta_{\mu,m}(\tau, z) \right)_{\mu \in \mathbb{Z}},
\]

where Diag denotes a diagonal matrix, the branch of the square root taken in (6) is the one with argument in \((-\pi/2, \pi/2]\), and \( \overline{\mu} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \) is the matrix with \( \mu \), \( \nu \) entry \( e_{2m}(-\mu \nu) \). Because of the transformation properties of the theta functions along with those of the Jacobi form, these vector-valued functions \( (h_\mu(\tau))_{\mu \in \mathbb{Z}} \) transform by the conjugate representations, i.e. (see [2] Thm 5.1)

\[
(\tau h_\mu(\tau + 1))_{\mu \in \mathbb{Z}} = \text{Diag} \left( e_{4m}(-\mu^2) \right)_{\mu \in \mathbb{Z}} \left( h_\mu(\tau) \right)_{\mu \in \mathbb{Z}},
\]

\[
\left( h_\mu \left( \frac{1}{\tau} \right) \right)_{\mu \in \mathbb{Z}} = \frac{\tau^{-k}}{\sqrt{2m}} \left( e_{2m}(\mu \nu) \right)_{\mu \in \mathbb{Z}} \left( h_\nu(\tau) \right)_{\nu \in \mathbb{Z}}.
\]

In particular, the above cited theorem states that there is an isomorphism between the space of Jacobi forms of weight \( k \) and index \( m \) and the space of vector-valued modular forms satisfying (7), and (8) that are bounded as \( \text{Im}(\tau) \to \infty \). This is the reason that we are able to focus on the \( \text{Sl}_2(\mathbb{Z}) \) invariance properties of the vector-valued modular form instead of the full Jacobi group \( \text{Sl}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \) invariance of the
Jacobi form. The theta functions $\theta_{n,m}(\tau, z)$ account for the elliptic invariance of the Jacobi form, and therefore we need only focus on the $SL_2(\mathbb{Z})$ invariance of the vector-valued modular form.

It is worth noting that the matrix in \( (8) \) is also known as the Schur matrix or the finite Fourier transform matrix and its properties (eigenectors, eigenvalues etc.) have been studied in [6],[10], and [14] among others.

Since \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) and \( \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) generate $SL_2(\mathbb{Z})$, the vectors of the $h_\mu(\tau)$ are vector-valued modular forms of weight $k - \frac{1}{2}$. That is, the vector of functions satisfy

$$
\left( h_\mu \left( \frac{a \tau + b}{c \tau + d} \right) \right)_{\mu(2m)} = (c \tau + d)^{k-\frac{3}{2}} \rho \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( h_\nu(\tau) \right)_{\nu(2m)},
$$

where \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) is an element of $SL_2(\mathbb{Z})$ and $\rho^{(a,b)}_{(c,d)}$ is a unitary matrix representation of a metaplectic cover of $SL_2(\mathbb{Z})$. The transformation formulas for the $h_\mu(\tau)$ provide relations among the component functions. We will use these properties to investigate how the different vector components are connected. In order to efficiently compute the matrix representations for general elements of $SL_2(\mathbb{Z})$ we use the results of [4], [8] (Chap. IX, sec. 3, (24)), among others. These papers give the matrix representations arising from more general theta functions and therefore we can take the conjugates to get the representations arising from the vector-valued modular forms. These authors primarily consider the representations arising from theta functions of quadratic forms of even rank, however the results are easy to generalize to a rank one quadratic form. These results for an element \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) of $SL_2(\mathbb{Z})$ with $c > 0$ may be summarized as

\begin{equation}
(9) \quad h_\mu \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^{k-\frac{3}{2}} \frac{1}{(2mc/i)^{\frac{1}{2}}} \sum_{\nu \mod 2m} \overline{\phi} \left( \mu, \nu, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) h_\nu(\tau),
\end{equation}

where

$$
\phi \left( \mu, \nu, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = \sum_{g \mod 2mc} e_{4mc} \left( ag^2 + 2g \nu + d \nu^2 \right)
$$

is a type of Gauss sum and the bar denotes complex conjugation. The transformation formulas for the case when the entry $c = 0$ are easily computed since these matrices are powers of \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) (or a power of \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \)) multiplied by \( -\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \). The case when $c < 0$ can also be treated by multiplying \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) the matrix \( -\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \). Applying the matrix \( -\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) to the function $h_\mu(\tau)$ yields that $h_\mu(\tau) = (-1)^k h_{-\mu}(\tau)$. Note that this last
fact implies that \( h_0(\tau), h_m(\tau) \equiv 0 \) if the weight is odd. Of particular use in this paper is the general fact that (see [1] or [3]) a Gauss sum of the form

\[
\sum_{j \mod \gamma} e^{\pi i \alpha j^2 + \beta j / \gamma}
\]

where \( \alpha, \gamma \) are relatively prime integers, is zero unless \( \beta \) is an integer and \( \alpha \gamma + \beta \) is even.

3. Transformations under congruence subgroups. The vectors of \( h_m(\tau) \) associated to a Jacobi form inherit their transformation properties from the Jacobi form and the theta functions. In fact the theta functions are vector-valued Jacobi forms of weight \( \frac{1}{2} \) and index \( m \) (in the sense that each individually is invariant under the elliptic transformation, and as a vector satisfy the \( SL_2(\mathbb{Z}) \) transformation properties).

In this section, the explicit transformation formulas under congruence subgroups will be established using the formulas given above. The congruence subgroups of interest here are

\[
\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \frac{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\mod n},
\]

\[
\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \frac{\begin{pmatrix} 1 & \ast \end{pmatrix}}{\mod n},
\]

\[
\Gamma_0^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \frac{\begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix}}{\mod n},
\]

\[
\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \frac{\begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}}{\mod n}.
\]

The lower left entries of these matrices will be crucial in the following calculations. One useful fact that will be used here is that

\[
\begin{pmatrix} a & b \\ cn & d \end{pmatrix} \in \Gamma_0(n) \leftrightarrow \begin{pmatrix} a & bc \\ n & d \end{pmatrix} \in \Gamma_0(n)
\]

along with generalizations to the other subgroups. These facts allow one to do the calculations more explicitly.

**Lemma 1.** If \( f(\tau, z) \) is a Jacobi form of weight \( k \) and index \( m \) for \( SL_2(\mathbb{Z}) \), then all of the associated functions \( h_m(\tau) \) are modular forms of weight \( k - \frac{1}{2} \) and on \( \Gamma(4m) \).
PROOF: This fact is easily shown using (9) and is essentially proven (for the theta functions) in [4] and [8]. In particular for a matrix \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) (using the useful fact above) in \( \Gamma(4m) \) the factors \( \phi(\mu, \nu, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)) \) are

\[
\sum_{g \equiv \mu \mod 2m} e_{16m^2} (ag^2 + 2g\nu + d\nu^2) = \\
\sum_{j=1}^{4m} e_{16m^2} \left( a(\mu + 2mj)^2 + 2(\mu + 2mj)\nu + d\nu^2 \right) = \\
\sum_{j=1}^{4m} e_{16m^2} (a\mu^2 + 2a\nu + d\nu^2) \sum_{j=1}^{4m} e_{16m^2} \left( (4m^2a j^2 + 4mj(a\mu + \nu)) \right).
\]

This last sum is zero unless \( a\mu \equiv -\nu \mod 2m \) and it is equal to \( 8m \) in this case (recall \( a \equiv 1 \mod 2m \) and \( h_{\mu}(\tau) = (-1)^k h_{-\mu}(\tau) \)). \( \square \)

**Lemma 2.** If \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(4m) \) then

\[
h_{\mu} \left( \frac{a\tau + b}{c\tau + d} \right) = (ct + d)^{k-\frac{1}{2}} \chi(d) e_{4m}(\mu^2ab) h_{-\nu}(\tau),
\]

where \( \chi(d) = \left( \frac{m}{d} \right)(-1)^{\frac{1-d}{4}} \).

Again this result is given in [9] (IX, sec. 4, Thm. 5, in terms of more general theta functions) but by taking conjugates and specializing to a rank one quadratic form, the result above follows. The two above results will be important for all that follows but we add a little more generality with the following lemma. In particular, we require a related result for \( \Gamma_0 \) groups of level dividing \( 4m \). This will give the decomposition of the representation as well as the vector-valued modular form on these groups.

**Lemma 3.** If \( m \) is square-free and \( t \) is an odd positive integer dividing \( m \), then for a matrix \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(4m/t) \),

\[
h_{\mu} \left( \frac{a\tau + b}{c\tau + d} \right) = \frac{(ct + d)^{k-\frac{1}{2}}}{\sqrt{2mc/t}} \sum_{\substack{\nu \equiv \mu \mod 2m \\ \nu \equiv -\mu \mod 2m/t}} c_{\mu,\nu} \left( \frac{a\nu}{c\nu} \right) h_{\nu}(\tau),
\]

where the constants \( c_{\mu,\nu}(\frac{a\nu}{c\nu}) = e_{(4m^2/t)}(a\mu^2 + 2\mu\nu + d\nu^2) \).

This result is not given in other works on the subject but it is an easy application of (9) so we give the proof here.
PROOF: In this case we assume that the lower left entry is $4m/t$ where $t$ is an odd positive integer dividing $m$. This assumption may be justified by the matrices used in [11] of the form
\[
\begin{pmatrix}
\beta & \beta \gamma - 1 \\
1 - \alpha \beta & \alpha + \gamma (1 - \alpha \beta)
\end{pmatrix}
\]
that are in $SL_2(\mathbb{Z})$ for any integers $\alpha, \beta, \gamma$ and may be chosen so that the lower left entry is $4m/t$.

\[
(11) \quad h_\mu \left( \frac{a \tau + b}{c \tau + d} \right) = \frac{(ct + d)^{k-\frac{1}{2}}}{(2mc/i)^{\frac{k}{2}}} \sum_{\nu \mod 2m} \phi \left( \mu, \nu, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) h_\nu(\tau)
\]

where
\[
\phi \left( \mu, \nu, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = \sum_{g \equiv \mu \mod 2m} \sum_{\nu \equiv \nu \mod 2m} e_{16m^2/it} \left( ag^2 + 2g\nu + dv^2 \right) =
\]

\[
\sum_{j \equiv 0 \mod 4m/t} e_{16m^2/it} \left( a(\mu + 2mj)^2 + 2(\mu + 2mj)\nu + dv^2 \right) =
\]

\[
e_{16m^2/it} \left( a\mu^2 + 2\mu\nu + dv^2 \right) \sum_{j \equiv 0 \mod 4m/t} e_{16m^2/it} \left( a4m^2j^2 + 4mj(\alpha \mu + \nu) \right).
\]

Note that the last sum is zero unless
\[
(4m^2a, 16m^2/t) = 4m(ma, 4m/t) = 4m^2/t(ta, 4) = 4m^2/t,
\]
divides $2m(\alpha \mu + \nu)$, i.e. unless $2m/t$ divides $(\alpha \mu + \nu)$ or $\nu \equiv -\alpha \mu \mod 2m/t$ and in this case the sum is trivially $4m/t$. \hfill \Box

The general case where one considers a general integer $t$ dividing $4m$ is similar. The only difficult part is determining the appropriate gcd. The result that we require and will show below is that for square-free index $m$ let $t' = (m, 2)$ and for $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(m/t')$

\[
(12) \quad h_\mu \left( \frac{a \tau + b}{c \tau + d} \right) = \frac{(ct + d)^{k-\frac{1}{2}}}{\sqrt{2mc/i}} \sum_{\nu \equiv \mu \mod 2m} c_{\mu \nu} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) h_\nu(\tau).
\]

4. Jacobi forms of square-free index. The above lemmas and the corresponding results for the functions $h_\mu(\tau)$ describe how these functions interact. Thus we can now express how the classes of Fourier coefficients depend on one another. For example we have the result:
THEOREM 1. If \( f(\tau, z) \) is a Jacobi form of square free index \( m \) and weight \( k \) then \( f \) is entirely determined by any one of the associated \( h_\mu(\tau) \) (except for \( \mu = 0, m \) in the case of odd weight, see the remark at the end of section 2). Moreover, any one of these functions can be used to generate the rest of the components and hence the Jacobi form.

This theorem can equivalently be stated as:

THEOREM 1'. If \( f(\tau, z) = \sum_{n, r} c(n, r)q^n z^r \) is a Jacobi form of weight \( k \) and square-free index \( m \), then \( f \) is entirely determined by the coefficients \( c(n, r) \) for \( r \) in some fixed congruence class modulo \( 2m \) (not equal to 0 or \( m \) in the case of odd weight) and the rest of the Fourier coefficients can be generated from this set.

It is clear and well-known that a Jacobi form is actually determined by a finite set of its Fourier coefficients. If one has an explicit basis for the finite-dimensional space of Jacobi forms of a given weight and index one could use linear algebra to compute the Jacobi form from a finite number of its coefficients. The proof given here is an explicit construction of the Jacobi form from an infinite set of coefficients, however the construction does not require a basis for the vector space of a given weight and index. In fact, the construction uses only the Jacobi form itself and the transformation laws that it satisfies. Thus if one knows part of the Fourier expansion of a Jacobi form (one associated vector component), the other parts of the Fourier expansion are forced by the transformation properties. Further the proof will contain a construction that explicitly computes the other parts of the Fourier expansion from one given component. In addition the proof will compute exactly on which subgroups of \( SL_2(\mathbb{Z}) \) the different vector components (or portions of the Fourier expansion) determine each other. For example, if \( p \) and \( q \) are primes dividing \( 2m \), and \( 2m \) is square free, then one can generate \( h_{pq}(\tau) \) from \( h_1(\tau) \) using actions from \( \Gamma_0(\frac{2m}{pq}) \) and this group is minimal in the sense that it cannot be generated from \( h_1(\tau) \) using elements from a proper subgroup of \( \Gamma_0(\frac{4m}{pq}) \). This implies that \( h_{pq}(\tau) \) is uniquely determined by \( h_1(\tau) \) provided the vector transforms over at least \( \Gamma_0(\frac{4m}{pq}) \); therefore this portion of the Fourier expansion of the Jacobi form is also uniquely determined by the portion corresponding to \( h_1(\tau) \) provided the Jacobi form transforms over at least \( \Gamma_0(\frac{2m}{pq}) \times \mathbb{Z}^2 \).

PROOF: Case I: Odd index.

We will show that all of the components \( h_\mu(\tau) \) can be uniquely determined from \( h_1(\tau) \). Because of the uniqueness of the other components, this implies the form is completely determined by any one of the components. The construction can also be done using the other
components but the details are more complicated (especially if starting with a component whose index is not relatively prime to $2m$).

Lemma 2 implies that, if $(a, b, c, d)$ is in $\Gamma_0(4m)$, then

$$h_{-a}(\tau) = h_1 \left( \frac{a\tau + b}{c\tau + d} \right) (c\tau + d)^{-\frac{k-1}{2}} \chi(d)e_{4m}(ab).$$

Therefore all of the vector components $h_{-a}(\tau)$ with $a$ relatively prime to $2m$ can be generated by acting on $h_1(\tau)$ with an appropriate matrix in $\Gamma_0(4m)$.

Then use Lemma 3 to generate the components $h_\nu(\tau)$ with $(\mu, m) = p$ for an (odd) prime $p$. Specifically, if $(a, b, c, d)$ is in $\Gamma_0(4m/p)$ then lemma 3 gives,

$$h_1 \left( \frac{a\tau + b}{c\tau + d} \right) (c\tau + d)^{-\frac{k-1}{2}} \sum_{\nu \equiv \mu \bmod 2m/p} \frac{\overline{c_1, \nu}(a, b, c, d)}{c_1, \nu(c, d)} h_\nu(\tau).$$

Exactly one of the components in the sum on the right hand side has its subscript divisible by $p$. This is clear because $(2m/p, p) = 1$ (m is odd and square-free) hence there is a unique $l$ in $\{0, 1, 2, \ldots, \frac{2m}{p} - 1\}$ with $pl \equiv -a \bmod 2m/p$.

$$h_{pl}(\tau) = \frac{(2mc/i)^{\frac{k}{2}}h_1 \left( \frac{a\tau + b}{c\tau + d} \right)}{c_1, pl(a, \frac{b}{c}, \frac{d}{c})} (c\tau + d)^{\frac{k-1}{2}} \sum_{\nu \equiv \mu \bmod 2m/p} \frac{\overline{c_1, \nu}(a, b, c, d)}{c_1, \nu(c, d)} h_\nu(\tau).$$

Since all of the other components are relatively prime to $2m/p$, and $p$ their indexes are also relatively prime to $2m$, these components can be computed explicitly using (13).

The other components $h_\nu(\tau)$ with $(\mu, m) = p$ can be found by applying lemma 2. Specifically, if $\mu = px$ with $(x, p) = 1$ then since $(2m/p, l) = 1$ there is a unique integer $a$ with $0 \leq a < \frac{2m}{p}$ such that $x \equiv -al \bmod 2m/p$ hence $\mu \equiv -alp \bmod 2m$. Thus for $(a, b, c, d) \in \Gamma_0(4m)$ we have

$$h_\mu(\tau) = h_{-alp}(\tau) = h_{4p} \left( \frac{a\tau + b}{c\tau + d} \right) (c\tau + d)^{-\frac{k-1}{2}} \chi(d)e_{4m}((lp)^2ab).$$

For the components $h_\nu(\tau)$ such that $(\mu, 2m) = 2$ a similar process may be used employing matrices in $\Gamma_0(m)$. The proof of lemma 3 is identical with $t = 4$ the only difference in the proof is that now the sum in lemma 3 is over indices $\nu$ with $a\mu \equiv -\nu \bmod m$. For example if $(a, b, c, d) \in \Gamma_0(m)$ with $a = 2$ then

$$c_{1, -2} \left( \frac{a}{c}, \frac{b}{d} \right) h_{-2}(\tau) = c_{1, m-2} \left( \frac{a}{c}, \frac{b}{d} \right) h_{m-2}(\tau) - \frac{(c\tau + d)^{-\frac{k-1}{2}}}{\sqrt{2mc/i}} h_1 \left( \frac{a\tau + b}{c\tau + d} \right).$$
Now $m - 2$ is odd since $m$ is odd and this component has already been determined (since the index is relatively prime to $2m$) and so we have computed $h_{-2}(\tau)$. Now the other vector components $h_\mu(\tau)$ with $(\mu, 2m) = 2$ can be generated by acting on $h_{-2}(\tau)$ with appropriate elements from $\Gamma_0(4m)$ and lemma 2 as above.

Once these are computed we proceed with two primes $p, q$ dividing $2m$. Again use lemma 3 and for some element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(\frac{4m}{pq})$ we have that

$$h_1 \left( \frac{at + b}{ct + d} \right) = \frac{(ct + d)^{k-\frac{1}{2}}}{(2mc/i)^{\frac{1}{2}}} \sum_{\nu \equiv -a \mod \frac{2m}{pq}} c_{1, \nu} \begin{pmatrix} a & b \\ c & d \end{pmatrix} h_\nu(\tau),$$

Exactly one term on the right hand side includes a vector component with index divisible by $pq$ (call this component $h_{pq}(\tau)$), since $(\frac{2m}{pq}, pq) = 1$. All of the other terms have a vector component with index $\mu$ such that $(\mu, 2m) = 1, p, q$ and these components have been determined above so we can solve for $h_{pq}(\tau)$ in terms of the known components. Then one can act on this component by elements of $\Gamma_0(4m)$ to produce all of the other vector components $h_\mu(\tau)$ with index $(\mu, 2m) = pq$ as in lemma 2. The components with $(\mu, 2m) = 2p$ for an odd prime $p$ can be found by acting on $h_1(\tau)$ by elements in $\Gamma_0(m/p)$ similar to the earlier calculation.

The process continues in this manner generating all vector components $h_\mu(\tau)$ where $(\mu, 2m)$ is a product of three primes (one of which may be even). This is done by acting on $h_1(\tau)$ with elements of $\Gamma_0(\frac{4m}{pq^r})$ where $p, q, r$ are three of the factors of $2m$, with the assumption that $p = 4$ is used to generate the components with even index. At the final stage all of the components with $(\mu, 2m) = 2m/p$ have been generated by acting on $h_1(\tau)$ with the elements from $\Gamma_0(p)$ for all primes $p$ dividing $m$ and $p = 4$ for the component $h_m(\tau)$. So all of the components except $h_0(\tau)$ have now been generated and we can determine $h_0(\tau)$ by acting on $h_1(\tau)$ with the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The sum that arises from (8) for $h_1(\tau)$ involves all of the components $h_\mu(\tau)$ $\mu = 0, (2m - 1)$. All of the components except for $h_0(\tau)$ have already been computed and therefore we can solve for $h_0(\tau)$ from this equation.

**Case II: Even index.**

To handle even indexes it is necessary to change the above method slightly. The only difference occurs in generating the vector components with even index. These components can be found using the matrices in $\Gamma_0(m/2)$. The action of these matrices is not considered in lemma 3 so we need to do a separate calculation. As noted above
the only relevant aspect is that we must determine greatest common
divisor of the coefficient of \( j^2 \) and the denominator in the Gauss sum.
Most of the proof of lemma lemma 3 is identical in this case so in computing the coefficients \( \phi(\mu, \nu, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \) we still need to evaluate the sum
\[
\sum_{j \mod \frac{4m}{t}} e^{16m^2 j^2 / \lambda} \left( a^2 4m^2 j^2 + 4mj(a\mu + \nu) \right)
\]
for the case where \( t = 8 \). Note that \((4m^2a, 16m^2/\lambda) = 4m(4m/\lambda) = 2m^2 \) and for this sum to be non-zero we need that \( 2m^2 \) divides \( 2m(\mu + \nu) \) or that \( \mu \equiv -\nu \mod m \).

For example if \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is an element of \( \Gamma_0(m/2) \) then
\[
h_1 \left( \begin{pmatrix} \tau + \frac{1}{\lambda} \\ \tau + \frac{1}{\lambda} \end{pmatrix} \right) = \frac{(\tau + \frac{1}{\lambda})^{-1} \sqrt{2m/\lambda}}{(c_1, -a^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}) h_{-a}(\tau) + c_{1, m-a}^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} h_{m-a}(\tau)} \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - e_{4m}(a^2) \right) = ch_{m-a}(\tau)
\]
(note that the operator on the right means to apply \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) to the variable \( \tau \) then subtract the constant times the original sum of functions) for an explicitly computable (non-zero) constant \( c \). A similar method can also be used to solve for the other component \( h_{-a}(\tau) \) by replacing the factor of \( e_{4m}(a^2) \) by \( e_{4m}(a - m)^2 \). Note that one of these components has index divisible by two and the other has index divisible by four (since \( m \equiv 2 \mod 4 \)). Once components \( h_{\nu}(\tau) \), and \( h_{\mu}(\tau) \) with \( (\nu, 2m) = 2 \) and \( (\mu, 2m) = 4 \), then all of the other components with similar indexes can be generated as above using elements of \( \Gamma_0(4m) \).

Then the computation proceeds as above constructing all of the vector components \( h_{\nu}(\tau) \) where \( (\nu, 2m) \) is a product of two odd primes \( pq, 2p, \) or \( 4p \), then a product of three factors of this form and so on. Using this method all of the even components except \( h_0(\tau) \) and \( h_m(\tau) \) can be found (recall these are zero in the case of odd weight). Finally to compute \( h_0(\tau) \) and \( h_m(\tau) \) we use (8) for \( h_1(\tau) \) and \( h_2(\tau) \). Both inversion formulas involve both \( h_0(\tau) \) and \( h_m(\tau) \) plus the other already computed components, however the inversion formula for \( h_1(\tau) \) contains \( h_0(\tau) - h_m(\tau) \) whereas the same formula for \( h_2(\tau) \) contains \( h_0(\tau) + h_m(\tau) \) and so we can solve for each component.
To see the uniqueness note that if two forms have the same associated \( \mu \)th vector component (except for \( \mu = 0, m \) in the case of odd weight), then the difference of the forms would be a Jacobi form of the same weight and index with a \( \mu \)th component zero. Examining the process used to find this component, it is either found using an element of \( \Gamma_0(4m) \) or \( \Gamma_0(4m/\ell) \). If it was found using \( \Gamma_0(4m/\ell) \) then the calculations above show it as the sum of some of the other components. These components have different Fourier expansions (see (4)) and therefore cannot cancel. The components generated by \( \Gamma_0(4m) \) similarly cannot be zero unless the component used to generate them was identically zero. For indexes \( 0, m \) the construction is a bit different. In the case of odd index, these components were computed directly from the other components and hence must have been uniquely determined. In the case of even index, if one of these components is identically zero then either \( h_1(\tau) \) or \( h_2(\tau) \) is zero and thus the rest of the components must also be zero. Therefore if one of these components is zero then the form must be identically zero. \( \square \)

Note that to construct the Jacobi form from a component that does not have a subscript relatively prime to \( 2m \) is more difficult and far more complicated from a computational perspective. It involves taking combinations of the \( h_\mu[A] \) for various matrices \( A \) in \( SL_2(\mathbb{Z}) \), separating out the square classes modulo \( 4m \) (using the translational invariance or Fourier expansions) and producing enough linear combinations so that the systems can be solved explicitly.

### 4.1 Jacobi forms of general index.

The case when the index of the Jacobi form is divisible by squares is more complicated, and the proofs seem much more involved. It is clear that if the index is not square free then there is an obstruction to generating the Jacobi form from one of its associated vector components. This can be seen by considering the operator \( U_l \) given in [2]. The operator acts on a Jacobi form of weight \( k \) and index \( m \) by

\[
(15) \quad f(\tau, z) U_l = f(\tau, lz)
\]

and this new function is a Jacobi form of weight \( k \) and index \( ml^2 \). If

\[ f(\tau, z) = \sum_{n, r} c(n, r) q^n \xi^r \]

then

\[
f(\tau, lz) = \sum_{n, r} c(n, r) q^n \xi^{lr} = \sum_{n, r} c'(n, r) q^n \xi^r
\]
and so \( f(\tau, lz) \) is a form whose Fourier coefficients \( c'(n,r) \) are zero unless \( l \) divides \( r \). Hence the associated vector-valued modular form components \( h_\mu(\tau) \) are identically zero unless \( l \) divides \( \mu \). It is clear that such a form cannot be determined by \( h_1(\tau) \) or other components \( h_\mu(\tau) \) with \( (\mu, 2m) = 1 \) (since it is identically zero though the form is not). In addition adding \( f(\tau, lz) \) to another form \( g(\tau, z) \) of index \( ml^2 \) (and the same weight) will alter only some of the associated vector-valued components (those with index \( \mu \) such that \( (\mu, 2m) = l \)). Thus a single component \( h_\mu(\tau) \) with \( (\mu, 2m) = 1 \) cannot uniquely determine the form even if it is non-zero.

In order to uniquely determine the Jacobi form in the case the index has square factors, one would need to specify a set of the vector components where the set must include one component \( h_\mu(\tau) \) with \( n|\mu \) for each \( n^2|m \) \((n > 1)\), along with a single component \( h_\nu(\tau) \) with \( (\nu, 2m) = 1 \). This number of components should be minimal since the components \( h_{an}(\tau) \) with \( a = 0, 1, \ldots, 2m/n^2 \) can be altered by adding a form of index \( 2m/n^2 \) that was raised to index \( 2m \) by the operator \( U_n \) (this assumes though that there are forms of index \( 2m/n^2 \)). This would not affect any of the vector components \( h_\nu(\tau) \) with \( n \not| \nu \). To show this number of vector components is sufficient would involve computing all of the vector components by similar calculations as in the square-free index case. To prove this would be very difficult to write down explicitly since the calculation would be different for cases where \( p^2|m, p^3|m, \) etc. and one would also need to deal with powers of two separately. Then one should be able to prove the form is uniquely determined by computing differences of the original Jacobi form and the constructed one and reducing the difference with a lowering operator that acts as an inverse to \( U_t \) (the operator is defined in [11] and a similar operator is defined in [12]).

The statements and constructions above show how the different vector components associated to a Jacobi form are related through the \( SL_2(\mathbb{Z}) \) invariance. For certain Jacobi forms we can prove more. In [2] (sec.5), the authors define operators \( W_{m'} \) for each positive integer \( m' \) that exactly divides \( m \) (there are exactly \( 2^t \) such operators where \( t \) is the number of distinct prime factors of \( m \)). For each such \( m' \) there is an integer \( \zeta \) with \( \zeta \equiv 1 \mod 2m/m' \) and \( \zeta \equiv -1 \mod m' \), and the operator \( W_{m'} \) acts on the vector components by sending \( h_\mu(\tau) \) to \( h_{\zeta \mu}(\tau) \). The space of Jacobi forms has a basis of simultaneous eigenforms for all of these operators. For such an eigenform, it is shown that \( h_{\zeta \mu}(\tau) = \pm h_\mu(\tau) \) where the \pm factor depends on the particular Jacobi form (actually the operators divide the space of Jacobi forms into different subspaces). There is a different operator for each \( \zeta \) with
\( \zeta^2 \equiv 1 \mod 4m \) or for each \( \zeta \mod 2m \) such that \( \mu^2 \equiv (\zeta \mu)^2 \mod 4m \). So if the Jacobi form is an eigenform of all these operators then every \( h_\nu(\tau) \) with \( \nu^2 \equiv \mu^2 \mod 2m \) is equal to \( \pm h_\mu(\tau) \). For such an eigenform all of the components whose subscript is in the same square class modulo 4m are equal up to a factor of \( \pm 1 \). Hence these functions have even more structure in its Fourier expansion.

5. Jacobi forms on congruence subgroups. As an application of the above construction, given a Jacobi form of weight \( k \) and index \( m \), \( f(\tau, z) = \sum_{\mu(2m)} h_\mu(\tau) \theta_{\mu, m}(\tau, z) \) we consider forms

\[
f_M(\tau, z) = \sum_{\mu \in M} h_\mu(\tau) \theta_{\mu, m}(\tau, z)
\]

where \( M \) is some set of congruence classes modulo \( 2m \). The above constructions allow one to determine on which subgroups these functions transform. Similarly, we can study the structure of the Fourier expansion of a Jacobi form that transforms on \( \Gamma \ltimes \mathbb{Z}^2 \) where \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \).

We first note that the association between Jacobi forms and vector-valued modular forms can be easily generalized to congruence subgroups of \( \text{SL}_2(\mathbb{Z}) \). That is, a Jacobi form will transform on the semidirect product of a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) and \( \mathbb{Z}^2 \), if and only if the associated vector-valued modular form transforms under generators of the congruence subgroup (the transformation formulas from the restriction of the representation of \( \text{SL}_2(\mathbb{Z}) \) given above (7), (8) to that congruence subgroup). The proof of this fact is identical to the one given for \( \text{SL}_2(\mathbb{Z}) \) in [2] and is omitted here.

**Proposition 1.** Let \( f(\tau, z) = \sum_{\mu \mod 2m} h_\mu(\tau) \theta_{\mu, m}(\tau, z) \) is a Jacobi form of weight \( k \) and square-free index \( m \) on \( \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \). Define \( M(\delta) = \{ \mu \mod 2m \mid (\mu, 2m) = \delta \} \) and \( \overline{M(\delta)} = \{ \mu \mod 2m \mid \mu \not\in M(\delta) \} \), then \( f_M(\delta, \tau, z) \) and \( f_{\overline{M(\delta)}}(\tau) \) are Jacobi forms on \( \Gamma_0(2m) \ltimes \mathbb{Z}^2 \). Furthermore, this is the largest group (of the form \( \Gamma \ltimes \mathbb{Z}^2 \) for a congruence subgroup \( \Gamma \)) on which these functions transform.

**Proof:** This follows from the construction in the theorem above. Any component \( f_{M(\delta), \mu}(\tau) \) will generate all of the other components \( h_\mu(\tau) \) involved in \( f_M(\delta, \tau) \). For example if \( \nu \equiv -a \mu \mod 2m \) where \( (a, 2m) = 1 \) (so that \( (\nu, 2m) = (\mu, 2m) = \delta \)) then if \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2m) \) we have

\[
f_\nu(\tau) = \frac{c + d}{2mc \sqrt{i}} f_{c, \mu} \begin{pmatrix} a \\ b \\ c & d \end{pmatrix} f_\mu \begin{pmatrix} a \mu + b \\ c \mu + d \end{pmatrix}.
\]
This is shown identically to the proof in lemma 3 with \( t = 2 \). Therefore, the vector-valued modular form made of these components will transform on \( \Gamma_0(2m) \) and hence the Jacobi form will transform on \( \Gamma_0(2m) \ltimes \mathbb{Z}^2 \). In addition, since any component can be used to generate the others using elements of \( \Gamma_0(2m) \) the form is uniquely determined by the component. The form cannot transform on a larger group since elements from a larger congruence subgroup would allow one to generate components \( h_\nu(\tau) \) with \( (\nu,2m) \neq \delta \) from one of the non-zero components.

Then since the entire vector of components \( h_\mu(\tau) \) for \( \mu \mod 2m \) transforms for all of \( SL_2(\mathbb{Z}) \) it also transforms under \( \Gamma_0(2m) \). Since the space spanned by the components \( h_\mu(\tau) \) with \( (\mu,2m) = \delta \) is stable under \( \Gamma_0(2m) \), the complement of this space must also be stable under \( \Gamma_0(2m) \). This implies that the Jacobi form \( f_{M[\delta]}(\tau,z) \) is also a Jacobi form on \( \Gamma_0(2m) \ltimes \mathbb{Z}^2 \). This form cannot transform on a larger congruence subgroup because if it did, the by the same argument as above \( f_{M[\delta]}(\tau,z) \) would also transform on a larger group. \( \square \)

One can restate this in terms of the restrictions on the Fourier expansion of a Jacobi form on \( \Gamma_0(2m) \ltimes \mathbb{Z}^2 \). In particular if \( f(\tau,z) \) has Fourier expansion as above then

\[
f_{M[\delta]}(\tau,z) = \sum_n \sum_{r \in \mathbb{Z}, (r,2m) = \delta} c(n,r)q^n \xi^r
\]

will transform as a Jacobi form of the same weight and index on \( \Gamma_0(2m) \ltimes \mathbb{Z}^2 \). Further this group is maximal in that it will not transform on the semi-direct product of a larger congruence subgroup of \( SL_2(\mathbb{Z}) \) and \( \mathbb{Z}^2 \).

One can also use these techniques to show that if any of the associated vector components of a Jacobi form are identically zero then others must be as well. In particular we have:

**Proposition 2.** If \( f(\tau,z) = \sum_{\mu \mod 2m} h_\mu(\tau)\theta_{\mu,m}(\tau,z) \) is a Jacobi form of weight \( k \) and square-free index \( m \) on \( \Gamma \ltimes \mathbb{Z}^2 \) with \( \Gamma_0(4m) \subseteq \Gamma \) and \( h_\mu(\tau) \equiv 0 \) (i.e. the vector component is identically zero) for some \( \mu \mod 2m \) with \( (\mu,2m) = \delta \), then \( h_\nu(\tau) \equiv 0 \) for all \( \nu \mod 2m \) with \( (\nu,2m) = \delta \).

**Proof:** This also follows from the construction used in the theorem. The elements of \( \Gamma_0(4m) \) are used to generate all of the components \( h_\mu(\tau) \) for \( (\mu,2m) = \delta \) from any one of these components. Hence if one is zero then all of these components must be zero. \( \square \)
Actually this statement can be generalized to any Jacobi form on \( \Gamma \times \mathbb{Z}_2 \) with \( \Gamma_0^0(4m) \subseteq \Gamma \) since the only matrices used in the construction are elements of \( \Gamma_0^0(4m) \).

Similar constructions can be done for all subgroups \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) that contain \( \Gamma(4m) \). For brevity we only make general statements to explain the actions of the different subgroups instead of trying to state all of the results. That is, we state the extra conditions imposed on the Fourier expansion of the Jacobi form (or the vector components that must be nonzero) that occur in moving from smaller groups to larger congruence subgroups.

Note first that for a positive integer \( N \), \( \Gamma_1(N)/\Gamma(N) \) is isomorphic to \( \mathbb{Z}/\mathbb{N} \mathbb{Z} \) and the representatives are the matrices \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \) where \( b \) runs through \( \mathbb{Z}/\mathbb{N} \mathbb{Z} \). Therefore the difference between forms transforming on these two groups are the translational properties of the vector components (or equivalently on the shape of the Fourier expansions of the vector components).

Note also that \( \Gamma_0(N)/\Gamma_1(N) \) is isomorphic to \( (\mathbb{Z}/\mathbb{N} \mathbb{Z})^* \) and the representatives of the classes relate the vector component \( h_\mu(\tau) \) to those of the form \( h_{-a\mu}(\tau) \) where \( a \) is in \( (\mathbb{Z}/\mathbb{N} \mathbb{Z})^* \). So each non-zero vector component \( h_\mu(\tau) \) will force all the other components \( h_{a\mu}(\tau) \) to be non-zero as well (or if the set \( M \) as above contains \( \mu \) then it must also contain all of the \( -a\mu \) with \( a \) in \( (\mathbb{Z}/\mathbb{N} \mathbb{Z})^* \)).

6. Remarks. The above results were motivated about by a question about which of the vector components associated to a Jacobi form can be identically zero for a given weight and index. The above calculations give a fairly complete view of the interactions among the different vector components over different congruence subgroups. This gives some of the conditions for a Fourier expansion of a Jacobi form due to the invariance properties of the form. One advantage to this method is that it is completely constructive and also classical, requiring only properties of theta functions and evaluations of Gauss sums.

One can try to use this construction to explicitly compute the Fourier coefficients for one vector component in terms of another. The author’s calculations in this area led to rather curious relations (even for simple theta functions) where the coefficient was an infinite sum of the other coefficients multiplied by partial K-Bessel functions. However, there were no obvious conclusions or applications that could be drawn. One area of further study is to try and understand these sums and hopefully produce explicit relations among the different coefficients. This would
give a full understanding of the structure of a Fourier expansion for a
Jacobi form (or for general theta functions).

In addition, these relations between the different vector components
can also be examined pointwise throughout the upper half plane \( \mathfrak{h} \).
For example if it was known that one component has a zero at some
point in \( \mathfrak{h} \) this would force corresponding zeros for the other directly
related components. To make this precise assume \( f(\tau, z) \) is a Jacobi
form of weight \( k \) and index \( m \) that transforms on a group \( \Gamma \times \mathbb{Z}^2 \) with
\( \Gamma_0(4m) \subseteq \Gamma \). If the associated component \( h_\mu(\tau) \) has a zero at \( \tau_0 \in \mathfrak{h} \n\)
then all of the components \( h_{\mu}(\tau) \) for \( (a, 2m) = 1 \) all have a zero at \( \tau_0 \).
This is clear from (13).

On a given congruence subgroup, the values at the different cusps
for the vector components can also be related to each other using these
calculations. For example the value of \( h_\mu(\tau) \) at a cusp is directly related
to the value of \( h_{\mu}(\tau) \) from (13). In particular, \( h_\mu(\tau) \) is a cusp form
for \( \Gamma(4m) \) (i.e. vanishes at the cusps of the fundamental domain for
\( \Gamma(4m) \setminus \mathfrak{h} \) if and only if all of the other components \( h_{\mu}(\tau) \) for \( (a, 2m) = 1 \)
also are cusp forms.

The results given here could be used to give many maps from Jacobi
forms to half-integral weight modular forms, however the most im-
portant of these results are well presented in [12] as are the representation
theory aspects of these representations. The representations that arise
are representations of the finite groups \( SL_2(\mathbb{Z}/4m\mathbb{Z}) \) that have been
studied by numerous authors such as [5], [7], and [13] among others.
An excellent reference for the Gauss sum calculations is [1]. It is curious
to note that the same representation (actually a conjugate of this repre-
sentation) that appears in this situation occurs in physics, see [3] where
one also finds the evaluation of the Gauss sums that occur here. In
particular, this representation gives the classical linear maps of a wave-
function on a quantized torus. In particular the invariance properties
of these quantum wavefunctions satisfy identical invariance properties
to the vector-valued modular forms arising from Jacobi forms.

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